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AN EMPIRICAL LAW DESCRIBING HETEROGENEITY IN THE YIELDS OF AGRICULTURAL CROPS

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(With Seven Text-figures)

CONTENTS

1. INTRODUCTION AND REVIEW OF LITERATURE

IN seeking to improve the efficiency of field experiments the best size of plot has been the subject of much discussion. To determine this point many experiment stations have conducted "blank experiments" (sometimes called "uniformity trials"), in which the produce from an area of ground is harvested as a number of small plots. By combining data for adjacent units the yields from plots of different sizes and shapes can be determined and their variabilities compared. Although on some points the results from these experiments have been gratifyingly consistent, no method of determining from these experiments the best size of plot for any particular purpose has heretofore been suggested. Indeed, both on this and on other points the usual methods of presenting the results of blank experiments are often misleading.

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The usual method of evaluating the best size of plot for a given set of conditions is to refer to a figure like Fig. 2 A (p. 5) which presents data for a blank experiment with wheat using unit plots of $\frac{1}{2}$ sq. ft. The argument then is: from one to about four times the unit area reduction in variability for increasing plot size is rapid. For plot sizes greater than six times the unit there is little reduction in variability. Consequently the optimum size is somewhere in the region of four to six times the unit area.

This argument fails to observe the condition that the region of maximum curvature depends entirely on the scale of the co-ordinates against which the observations have been plotted. In Fig. 2 B the ordinate scale has been doubled and the abscissa scale has been reduced to one-sixth. It gives the curve which one might ordinarily have drawn if the unit plot had been 3 sq. ft. ($x=6$) instead of $\frac{1}{2}$ sq. ft. The conclusion would now be that the optimum size is somewhere in the region of 24–36 units. It will be shown that the relative rate of reduction in variability is the same for the whole range of plot sizes explored, and that consequently the above argument is fallacious.

An unscientific approach is again usually adopted when comparing the reduction of variability due to increasing size of plot with the reduction which is to be expected if the fertilities of adjacent areas of ground were uncorrelated, or which is attainable by random replication of unit plots. The latter condition is represented by the dotted lines in Figs. 2 and 3. This curve is useful for comparison, but since an hypothesis of zero correlation between adjacent areas cannot be regarded as probable (as was demonstrated by Harris, 1915, 1920) it is misleading to speak of it as "theoretically expected". Confusion thus raised seems to have led a recent writer, Wiebe (1935), into an opposite error. Recognizing that the disagreement of observed and so-called theoretical curves is due to correlation, he has calculated a correlation coefficient for each plot size and thence calculated back the "theoretical coefficient of variation when $r \neq 0$". He has not, however, calculated any theoretical curve in the algebraic sense of a locus of points conforming to some law, and the agreement between observed and calculated values tells nothing except that the arithmetic has been correct.

The usual method of representing the heterogeneity of a piece of ground is to construct a fertility contour map after the fashion of Fig. 1. This particular figure has been constructed from the yields of square plots, 2×2 ft., centred at distances of 1 ft. (i.e. adjacent $\frac{1}{2}$ sq. ft. units were combined to show "moving averages"). But in the great majority of blank experiments unit plots have been long and narrow, and heterogeneity maps have been constructed from such units. For comparison with these, other maps were constructed from the yields of oblong plots orientated in both directions across the field described in § 2. The contour lines thus obtained were found to run predominantly in the direction of the length of plot, whatever that direction might be. This was so because the long narrow plots failed to provide sufficient points showing where contours might be connected across the lengths of plots. This leads to an appearance of greater variability across the plots than along them, an appearance which may often be, as in the present instance, wholly fictitious.

No satisfactory quantitative measure of soil heterogeneity has heretofore been devised. Harris (1915) proposed using the intraclass correlation coefficient of yields from adjacent areas as a "coefficient of heterogeneity". But although numerous workers have taken the trouble to evaluate such coefficients for their data it does not appear to serve any other purpose than to demonstrate that the fertilities of adjacent areas are correlated. This is demonstrated equally well, if not indeed better, and with much less labour, by a figure such as Fig. 2.

A number of blank experiments have shown that in some fields the variation of fertility is greater in one direction than in another. In such fields the shape of plot requires to be considered, since plots having their lengths in the direction of greater soil variability are less variable than similar plots orientated in a direction of lesser variability. In such fields the arrangement of differently shaped plots to form blocks is critical, and some apparently anomalous results reported in the literature can be ascribed to the way in which the data have been grouped in arbitrarily arranged blocks.

With regard to shape of plot as an independent factor affecting variability the most extensive discussion has been given by Christidis (1931). Following a theoretical discussion corroborated by a certain amount of observational data this author concluded that "in no case can square plots be more uniform than long narrow ones". Unfortunately the premises whence this deduction emanates contain a limitation to the arrangement of plots within blocks which is not consonant with practice and which appears to have been overlooked. It can be shown also from published data, for example, Day (1920); Wood & Stratton (1909), that long narrow plots may be more or less variable than square ones depending upon their orientation. Nevertheless, it does appear that long narrow plots tend to be on the average less variable than square plots, a finding which agrees with the observation that adjacent areas tend to be more closely correlated than more distant areas.

2. A blank experiment with small plots of wheat

Results of previous blank experiments have consistently shown that for a given area of ground maximum accuracy can be attained by the use of the smallest plots that are consistent with other requirements. In analytical yield experiments much of the labour is directly proportional to the "test area", and this may, if required, be reduced to the area occupied by a single plant per plot. In such extreme circumstances

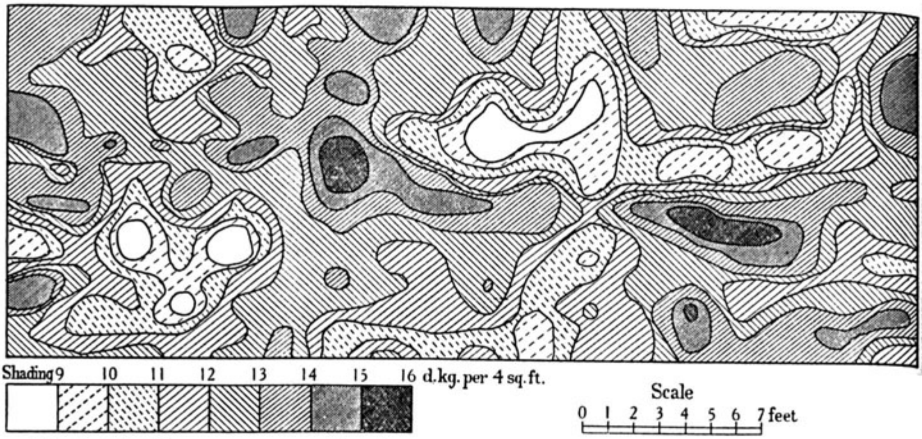

Fig. 1. Fertility contour map (based on moving averages for yield of grain from areas 2 ft. sq.)

the ratio of guard to test area will be unfavourable but may be counterbalanced if the gain in efficient use of the test area is sufficiently great. Nevertheless, a single plant per plot is unlikely to be efficient, and to obtain information as to the lower limit of plot size for which the above principle might apply a small blank experiment was conducted at Canberra in 1934.

Description of the experiment. Wheat of the variety Waratah was sown with a Woodfield dibber[1] which deposited the seeds uniformly 2 in. deep and 2 in. apart in rows 6 in. apart. At harvest (December 1934) four rows at each side and 1 ft. at each end of a row were discarded to avoid border effects. The remainder—15 ft. (thirty rows) by 36 ft.—was

[1] The Woodfield dibber was invented and is manufactured by C. E. Woodfield at the New Zealand Wheat Research Institute, Christchurch. It has been described in the First Annual Report of the Institute (1930), p. 6.

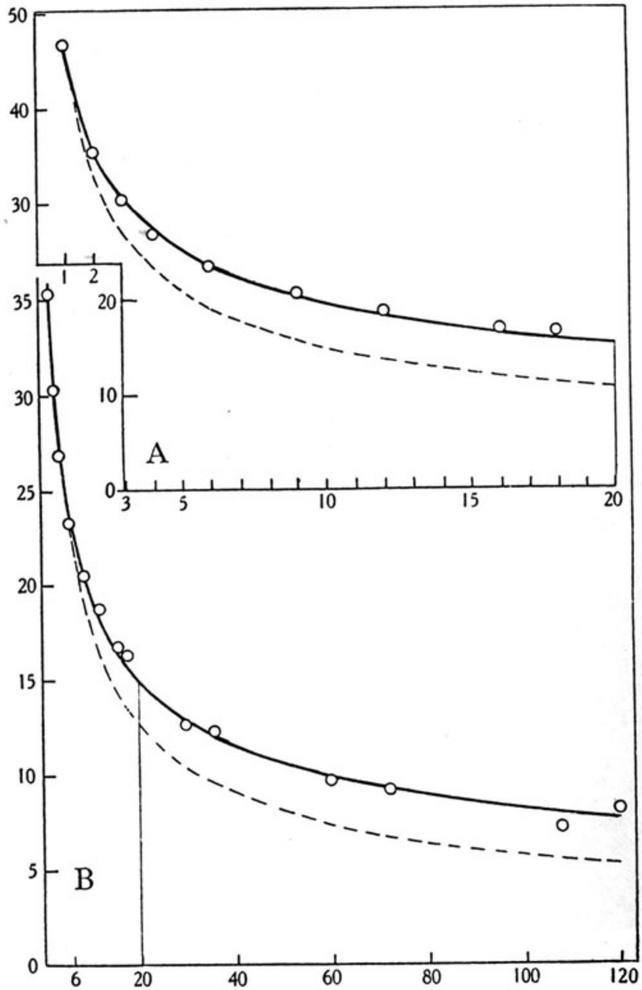

Fig. 2. Plot size and standard deviation per plot. Ordinate: standard deviation per plot (s_x) in decigrams per $\frac{1}{2}$ sq. ft. Abscissa: size of plot (x) in units of $\frac{1}{2}$ sq. ft. The dotted line indicates the reduction in standard deviation obtainable by combining units (or in $2B$ groups of 6 units) which are not correlated. The solid line represents the equation

$$s_x = \sqrt{(2137/x^{0 \cdot 746})}.$$

Each point represents the average variance of all shapes of each size which could be fitted into the total area (Table II).

harvested as 1080 plots of ½ sq. ft. (1 ft. length of row).[1] The average number of plants per plot was 5·02 (from six seeds sown).

The full data for yield of grain and for numbers of ears per foot length will be lodged in the archives at the Natural History Museum at South Kensington and in the Division of Plant Industry at Canberra.

The soil heterogeneity of the area was found to be patchy and equally variable in all directions. This is shown by a fertility contour map (Fig. 1) constructed from the moving averages of plots 2 ft. sq., and is corroborated by the mean squares between columns and between pairs of rows (taken in pairs to give a width of 1 ft. and so be comparable with the columns) being similar, namely 4739 and 4560.

The distribution of yields from unit plots was symmetrical and differed from a normal distribution only very slightly in the direction of a flat-topped curve[2]—g_2 negative but small (Fisher, 1932, § 14).

Variances of plots of several sizes and shapes are given in Table I.

Shape of plot. Table I shows that there was no consistent change of variability relative to shape of plot. For any particular size the differences are nowhere statistically significant.

Size of plot. Fig. 2 shows that the reduction of variability with increasing plot size is similar to that which has been observed in all blank experiments previously reported, and that such reduction is less than could be obtained by equivalent random replication.

The regression of variability on plot size can be more easily interpreted when the observations are plotted on double logarithmic paper (Fig. 3). It becomes, in these conditions, linear. This means that the *relative* reduction in variability for a *relative* increase of plot size is similar throughout the range observed, that, say, doubling the plot size always results, on the average, in the same proportional reduction in variability.

The relationship can be described by an equation of the form

$$\log V_x = \log V_1 - b' \log x,$$

where V_x is the variance of yield per unit area for plots of x units of

[1] This arrangement was not ideal. Analysis of results for shape of plot would have been considerably simplified if unit plots had been square. Further, to facilitate the formation of numerous shapes and sizes of plots while still covering the same total area of ground, the numbers of unit plots in each direction should be a multiple of 12. These observations must have been made by all workers who have had occasion to analyse one of these experiments, but it is a practical detail which is nearly always overlooked in the preliminary planning.

[2] This was determined graphically using probability graph paper described by K. G. Karsten, *Charts and Graphs*, chap. 40. New York: Prentice-Hall Inc., 1925.

Table I. *Variances of plots of different sizes and shapes, Canberra*, 1934

Plot size (x) in units of $\frac{1}{2}$ sq. ft.	Plot shape: length in ft.		Variance (V_x) for yield per sq. ft. in decigrams	Degrees of freedom	Area used Rows-feet (total area =30·36)
	Across rows	Along rows			
1	½	1	2201	1079	Total
2	½	2	1220	539	,,
	1	1	1272	539	,,
3	½	3	863	359	,,
	1½	1	976	359	,,
4	1	2	721	269	,,
6	½	6	524	179	,,
	1	3	523	179	,,
	1½	2	582	179	,,
	3	1	562	179	,,
9	½	9	413	119	,,
	1½	3	426	119	,,
	4½	1	450	29	27·10
12	½	12	334	89	Total
	1	6	383	89	,,
	1½	4	386	89	,,
	2	3	326	83	28·36
	3	2	310	89	Total
	6	1	376	71	24·36
15	½	15	404	29	30·15
16	2	4	281	62	28·36
18	½	18	276	59	Total
	1	9	272	59	,,
	1½	6	282	59	,,
	3	3	241	59	,,
	4½	2	214	53	27·36
	9	1	275	35	18·36
30	15	1	158	35	Total
36	½	36	102	29	,,
	1	18	178	29	,,
	1½	12	181	29	,,
	2	9	171	27	28·36
	3	6	155	29	Total
	4½	4	159	26	27·36
	6	3	146	23	24·36
	9	2	167	17	18·36
60	2½	12	98	17	Total
	5	6	77	17	,,
	15	2	102	17	,,
72	1	36	63	14	,,
	3	12	102	14	,,
	6	6	96	14	24·36
108	1½	36	52	9	Total
120	5	12	32*	8	,,
	15	4	98*	8	,,

* Difference is not significant. $z = 0{\cdot}550 \pm 0{\cdot}353$.

area. Since the variances have been estimated from varying numbers of plots it is desirable that, when fitting a regression, each point should be weighted inversely as its variance.[1] In fitting a linear regression to these

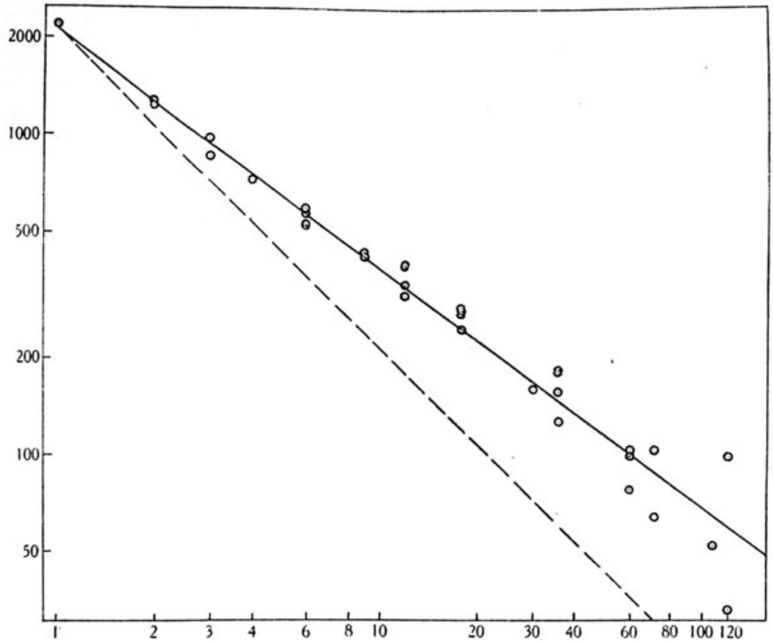

Fig. 3. Logarithmic relationship between variance per plot and plot size. Ordinate: logarithm of variance per plot ($\log s_x^2$). Abscissa: logarithm of size of plot ($\log x$). Dotted line as in Fig. 2 (but passing through the value for s_1^2 estimated from the regression instead of through the observed value). (Only plot sizes which fit exactly into the total area are shown.)

data the observations of the dependent variates are logarithms of variances. To a first approximation the variance of the logarithm of a variance may be taken as

$$\{\delta\,(\log_e s^2)\}^2 = \left\{\frac{2s\,\delta s}{s^2}\right\}^2 = \frac{2}{n},$$

where $\delta s = s/\sqrt{(2n)}$ represents the standard deviation of s which may be considered small relative to the mean of s, $s^2 = V$, and n is the number of degrees of freedom upon which the estimate of variance is based.

[1] Statistical methods of curve fitting rest on the assumption that observations are independent. It is clear that this condition is not fully satisfied in the present case, but all we require is a measure of an empirical relation and tests of significance are of only secondary interest.

Weighting the observations, then, inversely in proportion to n the regression was found to be

$$\log_{10} V_x = 3{\cdot}3298 - 0{\cdot}7486 \log_{10} x,$$

or

$$V_x = \frac{2137}{x^{0{\cdot}749}},$$

where s_x or $\sqrt{V_x}$ is in decigrams per $\frac{1}{2}$ sq. ft., and x is in units of $\frac{1}{2}$ sq. ft. The apparent[1] standard error of the b' coefficient (0·7486) is $\pm$0·0132.

3. Review of blank experiments previously reported

In order to determine whether the equation representing the relationship between plot size and variability as observed in the Canberra blank experiment might be generally applicable other published data have been examined. Table II and Fig. 4 show the results for thirty-nine experiments. Excepting only a small number which had too few plots to give reliable results, or in which data have not been presented in a form adaptable for our purpose, Fig. 4 includes results of all published work available in Canberra libraries. Straight regression lines have been drawn in freehand, except that for WA and P the best fitting lines have been calculated.

It is clear that in general the regression of the logarithm of standard deviation on size of plot is substantially linear. Fields P, WA and WB show a tendency to curve upwards. For each of four experiments with sweet potatoes (FA to FH) two regressions are shown—one for single row (broken line) and one for multiple row plots. Since, however, the author (Thompson, p. 395) gives reasons for expecting that single-row plots should be more variable, and that parts of the same row may be correlated owing to peculiarities of technique with sweet potatoes, this has presumably nothing to do with soil heterogeneity. In fields HB, L, U, X and Y the results are affected by shape of plot, indicating that the soil is more variable in one direction than in another.

The values of the regression coefficients observed will be discussed in a later section after attention has been given to some theoretical matters. The extent to which variation of the coefficients may be due to differences in crop, soil or season cannot be satisfactorily determined from present data. Since errors of observation tend to lower correlation and the regression coefficient is a function of the correlation of adjacent areas, errors in technique—sowing, harvesting, threshing, weighing—and genetic variability of plants will tend to raise the observed value of b'

Table II. *Review of blank experiments giving estimates of* b *coefficients of heterogeneity* (*see Fig.* 4)

Crop	Location	Size of unit plots, sq. ft.	No. of unit plots in experiment	c.v. for plots of 1/40 acre	Mean yield cwt./acre	*b'*	*b*	Author	Journal reference	Reference, Fig. 4	Reference in Cochran's catalogue
Wheat	Victoria	272	160	5·9	12·2	0·44	0·33	Forster & Vasey	*Proc. roy. Soc. Vict.* (1928), **40**, 70	A	119
	Rothamsted	87	500	6·3	17·7	0·46	0·37	Mercer & Hall	*J. agric. Sci.* (1911), **4**, 107	B	124
	,,	0·82	1092	4·4	19·7	0·54	0·51	Kalamkar	*J. agric. Sci.* (1932), **22**, 783	—	123
	Nebraska	30	224	4·9	19·4	0·54	0·49	Montgomery	*Bull. U.S. Dep. Agric.* (1913), **269**	C	126
	Missouri	3·3	3100	3·7 6·9	11·5 11·5	0·80 0·58	0·80 0·47	Day	*J. Amer. Soc. Agron.* (1920), **12**, 100	U	118
	Australia	0·5	1080	1·7	25·9	0·74	0·72	Smith	This paper, Fig. 3	—	130
	,,	2·3	54	3·1	21·5	0·54	0·44	,,	Unpublished, *Ab*, Fig. 5	—	—
Wheat irr.	Idaho	15	1440	10·5	33·6	0·22	0·08	Wiebe	*J. agric. Res.* (1935), **50**, 331	Z	132
Maize	Arkansas	242	432	14·2	—	0·21	0·05	McClelland	*J. Amer. Soc. Agron.* (1926), **18**, 819	V	23
Sorghum for.	Texas	55	1920	7·9	80·5	0·42	0·35	Stephens & Vinall	*J. agric. Res.* (1928), **37**, 629	S	91
Mangolds	Rothamsted	218	200	4·1	586	0·48	0·39	Mercer & Hall	*J. agric. Sci.* (1911), **4**, 107	D	43
Beets	Minnesota	61	600	5·7	326	—	0·50	Immer	*J. agric. Res.* (1932), **44**, 649	WA	98
	,,	61	600	3·8	323	—	0·74	Immer & Raleigh	*J. agric. Res.* (1933), **47**, 591	WB	99
Potatoes	W. Virginia	35	290	18·9	84·0	0·29	0·18	Westover	*Bull. W. Va Dep. Agric.* (1924), **189**	EA	76
	,,	35	186	10·5	116·7	0·45	0·37	,,	,, ,,	EB	76
	,,	35	192	16·8	58·3	0·46	0·39	,,	,, ,,	EC	76
	,,	35	258	16·4	91·4	0·46	0·40	,,	,, ,,	ED	76
	,,	35	3309	25·1	56·4	0·32	0·26	,,	,, ,,	EE	76
	Saskatchewan	66	576	—	—	—	0·71	Kalamkar	*J. agric. Sci.* (1932), **22**, 373	X	72
	Ormskirk	73	618	—	—	—	0·59	Justensen	*J. agric. Sci.* (1932), **22**, 366	Y	71
Sweet potatoes	Maryland, 1929	45	1000	8·6	41·6	0·78	0·77	Thompson	*J. agric. Res.* (1934), **48**, 379	FA	75
	Maryland, 1930	45	2000	7·0	102·7	0·66	0·65	,,	,, ,,	FB	75
	South Carolina, 1929	45	1560	11·1	69·6	0·58	0·56	,,	,, ,,	FC	75
	South Carolina, 1930	45	1035	10·7	72·3	0·74	0·73	,,	,,	FD	75

Cotton irr.	**Sakha, 1921**	**1131**	**160**	**10·5**	—	**0·32**	**0·19**	**Bailey & Trought**	***Bull. Minist. Agric. Egypt*, Tech. Sci. Serv. (1926), 63**	**GA**	**25**
	Gemmeiza, 1921	1131	160	12·6	—	0·16	?	,,	,, ,,	GB	25
	Gemmeiza, 1922	1131	160	9·1	—	0·58	0·53	,,	,, ,,	GC	25
	Giza, 1921	452	154	25·2	—	0·34	0·21	,,	,, ,,	GD	25
	Giza, 1923	452	160	15·0	—	0·47	0·40	,,	,, ,,	GE	25
Cotton	Texas (Col. St.)	144	288	28·9	—	—	0·53	Reynolds *et al.*	*J. Amer. Soc. Agron.* (1934), **26**, 725	HA	32
	Texas (Chill.)	157	288	—	—	—	0·56	,,	,, ,,	HB	32
Soybeans for.	W. Virginia, 1925	20	1008	9·6	30·3	0·24	0·11	Odland & Garber	*J. Amer. Soc. Agron.* (1928), **20**, 93	IA	95
Soybeans seed	W. Virginia, 1926	20	1540	6·0	10·7	0·40	0·34	,,	,, ,,	IB	95
Pineapples	Hawaii (K 19)	—	24	—	—	0·42	0·20	Magistad & Farden	*J. Amer. Soc. Agron.* (1934), **26**, 631	KA	69
	Hawaii (K 1)	—	64	—	—	0·47	0·36	,,	,, ,,	KB	69
Natural pasture	Australia	22	720	{9·5 {15·8	8·7 8·7	0·63 0·37	0·60 0·29	Davies	*Bull. Coun. sci. industr. Res. Aust.* (1931), **48**	L	67
Oranges irr.	Riverside	3840	195	11·0	102	0·41	0·31	Parker & Batchelor	*Hilgardia* (1932), **7**, 81	M	65
	Arlington	484	1000	34·3	110	0·24	0·10	Batchelor & Reed	*J. agric. Res.* (1918), **12**, 245	NA	64
	Antelope H.	484	495	25·6	150	0·36	0·25	,,	,, ,,	NB	64
	Villa Park	484	240	79·8	198	0·44	0·32	,,	,, ,,	NC	64
Lemons irr.	Upland	576	364	26·2	183	0·43	0·34	,,	,, ,,	O	36
Walnuts irr.	Whittier	2500	280	57·5	13·5	0·52	0·47	,,	,, ,,	P	115
Apples irr.	Utah	484	224	32·1	244	0·54	0·47	,,	,, ,,	Q	4
Various crops	Quebec	174	175	—	—	0·34	0·20	Summerby	*Tech. Bull. MacDonald agric. Coll.* (1934), **15**	—	41
Theoretical curve if correlation of adjacent areas were zero, or reduction of error obtainable with random replication (b = 1·0).										T	—

b' is the slope of each regression estimated graphically from Fig. 4.

b is the value estimated as appropriate for an infinitely large field as described in the text. As defined in this paper the coefficients refer to regression of *variance* on plot size. Published data, which were plotted on log-log paper for the preparation of Fig. 4, give invariably standard deviations or coefficients of variability. The slopes of regressions as shown in Figs. 4 and 5 are thus one-half of the values in the table.

C shows data for 2 years combined, 1909—O, 1910—X.

U, b' = 0·80 for plots 5 ft. wide, marked ρ, 0·58 for all other shaped plots, single row plots marked O, intermediate shapes marked □.

WA and WB, error variance estimated for within blocks of 5 plots.

EA to ED, different strains of the same potato variety on adjacent areas of soil. Mean b' = 0·42, b = 0·34.

X, single row plots only within blocks of 6 plots.

Y, single row plots only within blocks of 5 plots.

FA to FD, multiple row plots. Single row plots shown as broken lines FE to FH (see text).

HA, within blocks of 12 plots.

HB, single row plots only within blocks of 12 plots.

L, plots 5 links wide marked ρ, b' = 0·63; other shapes O, b' = 0·37.

M, guarded plots, means of 6 years.

N to Q, unit plots = area occupied by a single tree. N to P in California, exact location given for reference in the publication concerned.

Irr. = irrigated. For. = forage.

as compared to a value characteristic of the soil alone. It would be of theoretical interest to obtain coefficients characteristic of the soils alone, but it seems at present impossible to disentangle the effects of these incidental variables. This is, however, of little practical consequence, since for purposes of experimental planning they should be included. They will presumably vary with different crops.

The regression

$$\log V_x = \log V_1 - b' \log x,$$

when given the appropriate constants, may be considered to describe the soil and plant heterogeneity of an observed field. In order to compare the variability of plot yields for different crops and soils the regressions observed in Fig. 4 have been converted to show the regressions of the coefficient of variability on plot size in square feet. The results are shown in Fig. 5. The corresponding coefficients of variability for a standard size of plot (1/40th acre) and mean yields are given in Table II. The crops fall roughly into three groups: wheat, mangolds, beets, soybeans and sorghum (forage) seem to be least variable; maize, potatoes, sweet potatoes, cotton and natural pasture are intermediate; and fruit trees are most variable.

4. Variance within blocks

In order to bring results deduced from blank experiments into consonance with modern experimental practice we require to consider the variability of different sized plots within blocks. At first sight this complicates the problem, since the possible combinations of varying sizes and shapes of plots with varying sizes and shapes of block are very great. But if shape may be ignored and we have a general law giving the variability of plots of varying size, then an expression for variance between, and thence for that within, blocks can also be derived.

In §§ 2 and 3 we have considered simply variance of plots over the whole area of the fields used for the experiments. But size of field is purely arbitrary, and any given field may be considered as but a single block of a larger field. It is advisable, therefore, that the terms should be more accurately defined to indicate the area over which a variance is measured. Let the variance of the mean yield per unit area, of plots of x units of area, over a block of m plots be $(V_x)_m$. Then the regression which has been empirically observed for a field having n plots is

$$\log (V_x)_{n/x} = \log (V_1)_n - b' \log x.$$

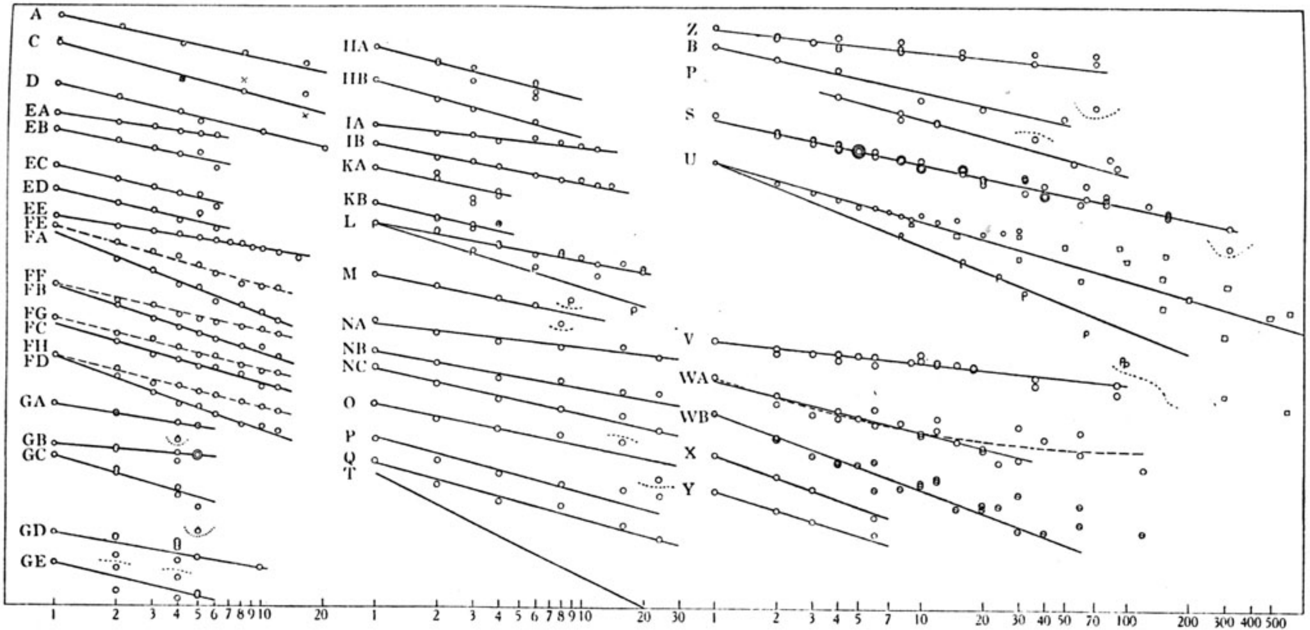

Fig. 4. Logarithmic relationship between plot size and standard deviation per plot from data from blank experiments reported in the literature. Ordinate: logarithm of standard deviation per plot ($\log s_x$). Abscissa: logarithm of size of plot ($\log x$). For key see Table II.

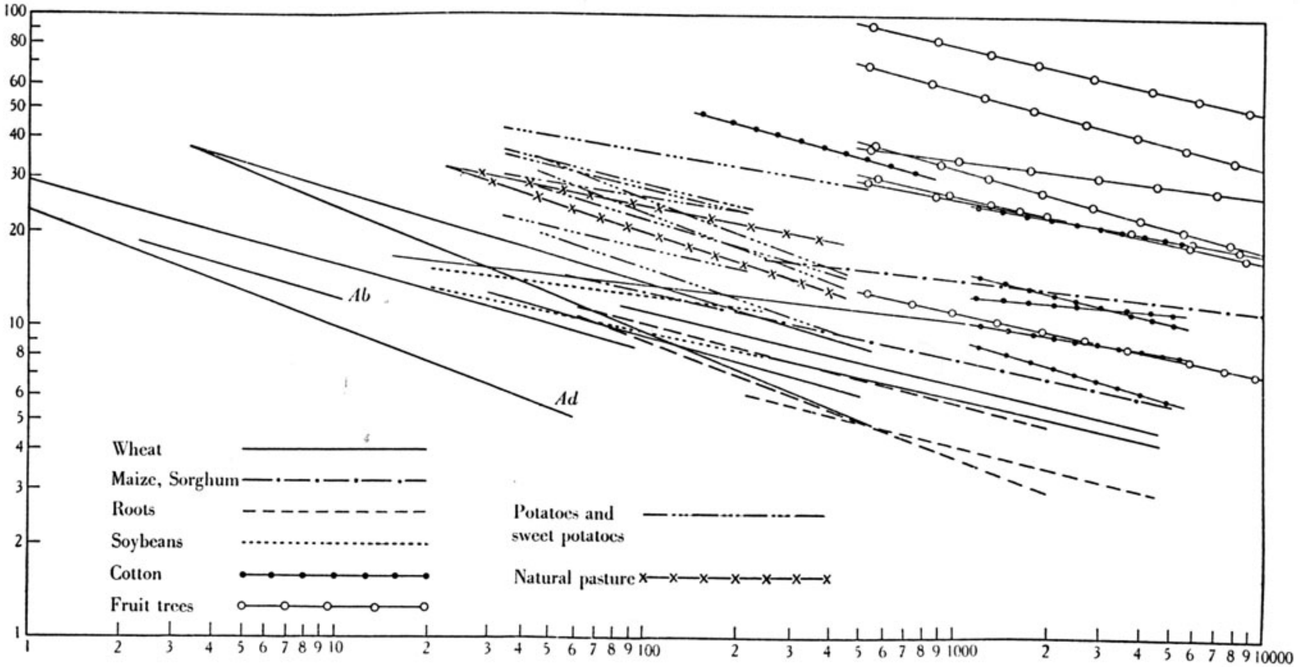

Fig. 5. Regressions of coefficient of variability on plot size in sq. ft. (Data in Table II.)
(43,560 sq. ft. = 1 acre.)

But the size of the field, n, is arbitrarily fixed, and if this regression is theoretically sound it should hold good for any alternative size of field which we may wish to consider as the area over which variances should be measured. But if n be varied the above regression becomes curved for any value other than that originally assigned. The regression is therefore inconsistent with the requirement that the law shall be unaffected by variation in the size of the field.

This difficulty is overcome if we postulate an infinitely large field of which an observed field may represent a single block and suppose that the law be

$$\log (V_x)_\infty = \log (V_1)_\infty - b \log x, \qquad \ldots\ldots(1)$$

or

$$(V_x)_\infty = \frac{(V_1)_\infty}{x^b}. \qquad \ldots\ldots(1a)$$

When $b=1$ this gives the ordinary formula for the variance of the mean of x independent units.

Since a block is merely a large-size plot the variance between blocks can be estimated from the same equation, and thence the variance of plots of x units within blocks of m plots is given by an analysis of variance, where λ tends to ∞, as follows:

	D.F.	Mean square
Between blocks	$(\lambda - 1)$	$m\,(V_{xm})_\lambda$
Within blocks	$\lambda\,(m-1)$	$(V_x)_m$
Total	$(\lambda m - 1)$	$(V_x)_{\lambda m}$

thus in the limit

$$(V_x)_m = \frac{m\,(V_x)_\infty - m\,(V_{xm})_\infty}{m-1} = \frac{m\,(1-m^{-b})\,(V_x)_\infty}{m-1} \qquad \ldots\ldots(2).$$

For given values of m and b, $(V_x)_m/(V_x)_\infty$ is a constant irrespective of x. Thus the association between neighbouring plots is of such a type that the variance per plot within a block of m plots bears the same ratio to the total variance per plot in an infinite field whatever the plot size; or, alternatively stated, the intraclass correlation between groups of m plots is the same whatever the size of plot.

On this hypothesis the variances recorded in §§ 2 and 3 should conform to a regression of the form

$$\log (V_x)_{n/x} = \log (V_1)_\infty - b \log x + \log \frac{n\,(n^b - x^b)}{n^b\,(n-x)}. \qquad \ldots\ldots(3)$$

This is a curve slightly concave downwards, but, as the curvature becomes appreciable only when x/n approaches one-tenth and in this region the variances are not well determined, the curvature could not be detected by the methods used in the preceding sections.

The value of b', however, estimated from a regression for a finite field will differ substantially from the equivalent b for the infinite field.[1] The most expeditious way to convert the observed b''s into equivalent b's is to determine the corrections for a series of values of b over the given range of x/n. This was done graphically. The magnitude of the corrections obtained is illustrated by the following table:

b	1·0	0·8	0·7	0·6	0·5	0·4
b' in range x/n from 0·001 to 0·01	1·0	0·804	0·710	0·617	0·528	0·443
b' in range x/n from 0·01 to 0·1	1·0	0·822	0·738	0·656	0·578	0·504
b	0·35	0·3	0·25	0·2	0·15	0·1
b' in range x/n from 0·001 to 0·01	0·403	0·364	0·326	0·291	0·257	0·226
b' in range x/n from 0·01 to 0·1	0·469	0·434	0·402	0·371	0·343	0·312

Equation (2) indicates that the expectation for the efficiency of a randomized block experiment having m plots per block relative to one with n plots per block is

$$\frac{(V_x)_n}{(V_x)_m}=\frac{n\,(m-1)\,(1-n^{-b})}{m\,(n-1)\,(1-m^{-b})}. \qquad \ldots\ldots(4)$$

As a guide to the effect on experimental error of changing the size of block, the curves for $(V_x)_m$, taking $(V_x)_\infty$ equal to unity, for various values of m and of b are plotted in Fig. 6. As an example of the use of these, suppose we wish to conduct an experiment with forty treatments and we want to know if it is worth while to adopt a complex design by which block size may be reduced (for example, by confounding interactions or "pseudo-factors" (Yates, 1935, 1936)) as compared to a simple lay-out using blocks of forty plots. If we expect the field to show a "b coefficient of heterogeneity" of 0·2 the expectation for relative efficiencies of 5, 10, 20 and 40 plot blocks is

$$\frac{0{\cdot}535}{0{\cdot}344}:\frac{0{\cdot}535}{0{\cdot}410}:\frac{0{\cdot}535}{0{\cdot}475}:\frac{0{\cdot}535}{0{\cdot}535}=1{\cdot}56:1{\cdot}30:1{\cdot}13:1,$$

and it will be worth considerable effort to be able to use small blocks. But if the b coefficients be about 0·7 the expectation of relative efficiencies is only

$$1{\cdot}12:1{\cdot}06:1{\cdot}03:1,$$

and it would be best to retain the simplicity attaching to the larger blocks with a simple design.

In a similar manner to the derivation of (2) above it can be shown

[1] It should be noted that an observed value of b' depends in part on the number of unit plots observed. This condition can be demonstrated from actual data, or may be deduced from an hypothesis that a linear logarithmic regression may be applicable for a finite field of given size, as well as from the law for a theoretical infinite field.

that where differential directional heterogeneity is not present (so that shape of block is unimportant) the error variance in an $m \times m$ latin square may be represented by

$$(V_x)_{m_2} = \left\{ \frac{m\,(1-m^{-b})}{m-1} \right\}^2 (V_x)_\infty .$$

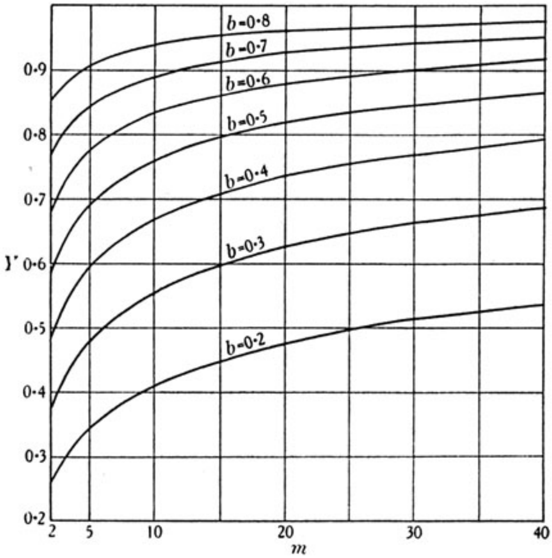

Fig. 6. To show reduction of error variance obtainable by subdividing an area into blocks for different values of the b coefficient of heterogeneity and of the number of plots per block (m). The ordinate, $Y = (1 - m^{-b})/(1 - m^{-1})$, shows the ratio of error variance within blocks to variance over an infinite area.

The frequency distribution of the adjusted b coefficients given in Table II is

Adjusted b	0·05–0·15	0·15–0·25	0·25–0·35	0·35–0·45	0·45–0·55	0·55–0·65	0·65–0·75	0·75–0·85	Total
Frequency	4	4	8	6	7	4	3	1	37

The mean is about 0·4, but since the limiting values of b are 0 and 1 and three-quarters of this range is covered by the observations, the diversity of conditions is very great.

5. The optimum size of plot

From equations ($1a$) and (2) the information per plot of size x in blocks of m plots is

$$\frac{1}{(V_x)_m} = \frac{(m-1)}{m\,(1-m^{-b})} \frac{x^b}{(V_1)_\infty}.$$

The cost per plot may be given by a linear regression

$$K_1 + K_2 x,$$

whence the cost per unit of information is

$$\frac{m\,(1-m^{-b})\,(K_1+K_2x)\,(V_1)_\infty}{(m-1)\,x^b}. \qquad \ldots\ldots(4)$$

This is minimum when

$$x = \frac{bK_1}{(1-b)\,K_2}. \qquad \ldots\ldots(5)$$

This provides a formula for estimating the most efficient size of plot for any given experiment. It is not affected by the number of plots, which may be determined either by the area available or by the number which will probably be necessary to reduce the error variance of treatments to any assigned figure.

6. Guarded plots

In experimental work it is customary to discard guard areas around all plots. The effect of this condition has been ignored in the preceding discussion.

Suppose that an area be divided into n plots side by side, of which each alternate plot may be taken as the guard area. Then if n is even we have the following analysis of variance:

	D.F.
Between experimental plots, E	$\frac{1}{2}n-1$
Between guard plots, G	$\frac{1}{2}n-1$
$\bar{E}-\bar{G}$	1
Total	$n-1$

The mean square for $\bar{E}-\bar{G}$ is likely to be smaller than the other two components, since it depends on the comparison of neighbouring plots, but if n be large the mean squares between E and between G will be only slightly larger than the mean square for the total, and may for practical purposes be regarded as approximately equal to it.

Extending this argument, we may say that if m guarded plots each with "test-area" x occupy the same total area as m' unguarded plots of the same area x, then the variance within blocks of the m guarded plots will be equal to $(V_x)_{m'}$. The procedure of § 5 may thus be simply modified so as to be applicable to guarded plots.

Thus, if a block consists of a single row of plots side by side, the end guards (area A) may be considered as lying outside the block. The area occupied by the side guards bears to the "test-area" x the ratio $B=(W-w)/w$ where w is the width of the test area and W the width of the test area plus guards. If K_g is the cost per unit area for handling guard areas, the cost per plot may be given by:

$$K_1 + K_2 x + K_g (A + Bx).$$

Thus, proceeding as before, the cost per unit of information is found to be a minimum when

$$x = \frac{b(K_1 + K_g A)}{(1-b)(K_2 + K_g B)}. \qquad \ldots\ldots(6)$$

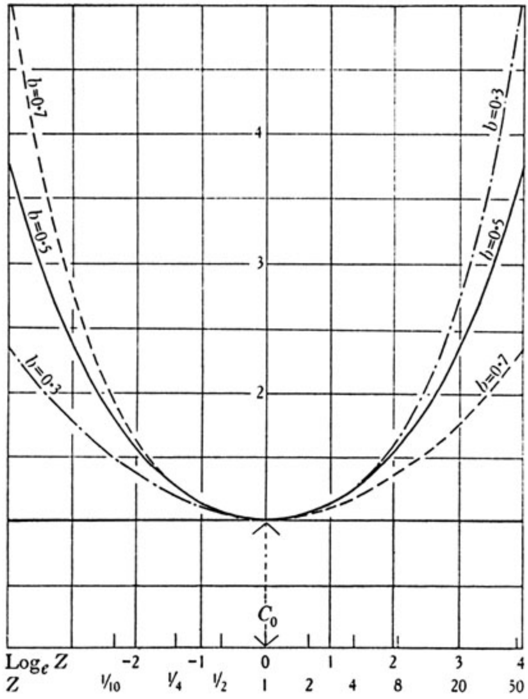

Fig. 7. To show increase of cost (or of error variance) if the optimum size of plot be not used. Ordinate: cost relative to cost using the optimum size of plot. Abscissa: logarithm of size of plot relative to optimum size.

7. Cost of using plot sizes other than the most efficient

Equation (4) gives the cost per unit of information as

$$\text{Const.}\ (K_1 + K_2 x)\, x^{-b},$$

which is minimum when

$$x = \frac{bK_1}{(1-b)\,K_2} = x_0.$$

If the most efficient size plot be not used the relative cost is

$$y = \frac{\text{cost}}{\text{min. cost}} = bz^{(1-b)} + (1-b)\,z^{-b} \qquad \text{......(7)}$$

$$= be^{(1-b)\ln z} + (1-b)\,e^{-b\ln z}, \qquad \text{......(8)}$$

where z is x/x_0, that is the ratio of the size of plot used to that size which is most efficient. The second form (8) is the more convenient both for calculation and to give a symmetrical scale for relative plot sizes. It is of interest to note that if $b=0{\cdot}5$, (8) is the equation of a catenary curve. Fig. 7 gives the curves for $b=0{\cdot}3$, 0·5 and 0·7. When $b=0{\cdot}5$ efficiency is

96 % if plots are double or half the optimum size.
80 % if plots are quadruple or quarter optimum size.
63 % if plots are eight times or an eighth optimum size.
47 % if plots are sixteen times or a sixteenth optimum size.

The value of b does not greatly affect these estimates in the region a quarter to four times the optimum size. Beyond this range if b is less than 0·5 it is more serious to have plots which are too large than to have plots which are too small, and vice versa.

8. Arithmetical example of estimating the most efficient plot size for a given experiment

In analytical yield trials it is required to count plant numbers, tiller numbers and ear numbers, weigh straw and grain, and estimate weight per grain. The b coefficient of the Canberra experiment field has been estimated to be about 0·75. Sowing with a Woodfield dibber fixes the rows (width of plot) at 4 ft., and rows are 6 in. apart. Six inches around each plot are discarded as guard. We thus have $A=4$ sq. ft., $B=0{\cdot}33$. Costs other than labour are relatively negligible, and it is assumed that the same class of labour is used throughout. From past experience it has been estimated that, on the average,

(1)	Preparing seed	requires	0·005	man hour per	sq. ft.
(2)	Sowing	,,	0·017	,,	sq. ft.
(3)	Counting plants at braird	,,	0·002	,,	sq. ft.
(4)	Counting tillers	,,	0·062	,,	sq. ft.
(5)	Observing earing and flowering dates	,,	0·033	,,	plot
(6)	Counting plants near harvest	,,	0·01	,,	sq. ft.
(7)	Cutting out guards	,,	0·001	,,	sq. ft.
(8)	Harvesting and weighing sheaf	,,	0·030	,,	sq. ft.
(8a)			0·025	,,	plot
(9)	Counting ears	,,	0·033	,,	sq. ft.
(10)	Threshing	,,	0·016	,,	sq. ft.
(10a)			0·016	,,	plot
(11)	Weighing grain	,,	0·016	,,	plot
(12)	Estimating average weight per grain	,,	0·06	,,	plot
(13)	Statistical analysis	,,	0·10	,,	plot

Whence

$K_1 = (5) + (8a) + (10a) + (11) + (12) + (13) = 0{\cdot}25$ man hour per plot.

$K_2 = (1) + (2) + (3) + (4) + (6) + (8) + (9) + (10) = 0{\cdot}18$ man hour per sq. ft.

$K_g = (1) + (2) + (7) = 0{\cdot}023$ man hour per sq. ft.

$$x = \frac{b\,(K_1 + K_g A)}{(1-b)\,(K_2 + K_g B)} = \frac{0{\cdot}75\,(0{\cdot}25 + 0{\cdot}023 \times 4)}{0{\cdot}25\,(0{\cdot}18 + 0{\cdot}023 \times 0{\cdot}33)} = 5 \text{ sq. ft.}$$

9. Discussion

The regression of plot variance on size is an empirical relationship for fields of a size normally considered in experimental work. Since it implies that adjacent areas are equally correlated irrespective of their size and this condition must sooner or later break down, the relationship cannot be extended indefinitely. It nevertheless appears to provide a method of evaluating approximately the average relative efficiencies of varying sizes of plots and of blocks. The wide range of values of b coefficients is a measure of the variation in types of soil heterogeneity. A similar degree of variation is demonstrated by the estimates of efficiencies of various randomized block and latin square experiments reported by Yates (1935). It is consequently impossible to forecast accurately the relative efficiencies of different arrangements for a field of whose heterogeneity little or nothing is known. One might perhaps anticipate that the b coefficients should be low for fields known to be heterogeneous and high for fields of fairly even fertility, but the absence of correlation between b and coefficients of variability in Table III indicates that any such assumption may not in fact be justified. So far as present evidence goes total variability and the manner in which varying fertilities are distributed appear to be two distinct features which must be separately considered in any quantitative measure of soil heterogeneity. It is to be noted, however, that the optimum plot size and the increase in information resulting from reduction in the number of plots per block is dependent only on the value of b.

Information about the persistence of similar values of b for a given field over several years and with different crops is required before we can say how far it may be worth while to determine coefficients appropriate for fields which are to be frequently used for experimental work. No evidence on this point is at present available,[1] except that if heterogeneity

[1] Data which might be used to test this question have been given by Summerby (1934) for oats, alfalfa and maize in 3–5 years, and by Garber *et al.* (*J. agric. Res.* (1926), **33**, 255–68) for oat hay and wheat grain in 2 years, and (*J. Amer. Soc. Agron.* (1931), **23**, 286–98) for maize, oats and wheat in 3 years. The present writer is, however, unable to devote to this problem the time required for the necessary calculations.

types were persistent one might perhaps expect the covariance of plots in successive years to be more useful than it has been found to be.

Since the regression of variance on plot size is a function of the correlation of adjacent areas, it appears theoretically inevitable that shape of plot should have some effect, since the correlation of ends of long narrow strips must usually be less than that at opposite sides of a square of equal area. The shapes of plots considered may therefore be expected to have some effect on the regression. The importance of plot shape is of course particularly accentuated in fields which show differential directional heterogeneity (e.g. U and L). In a field where the orientation of the variability is known and is reasonably persistent, long narrow plots with a particular orientation will obviously be called for. In such circumstances it might be worth while to make use of two b coefficients with assigned directions to describe the heterogeneity; one being applicable for estimates regarding size of plot, the other for considerations of block size. The effect of different shapes of plots or of blocks appears to be responsible for the fact that Yates (1936) using data from Parker & Batchelor obtained greater efficiency for small blocks than would be indicated by the regression M of Fig. 4 which is based on only four variances given by the original authors.

10. Summary

Using data from a blank experiment with wheat it was found that the regression of the logarithms of the variances for plots of different areas on the logarithms of their areas was approximately linear. A graphical review of variances, etc., reported in the literature for thirty-nine other blank experiments indicates that the results of most such experiments conform to the same law.

It is shown that the above law can be generalized (so as to be applicable to any size of field) by applying a certain adjustment to the regression coefficient b', so as to give a modified coefficient b applicable to an "infinite" field.

From this generalized relationship there has been deduced an expression ((4), p. 16) to indicate average relative efficiencies to be expected for randomized block experiments with varying numbers of plots per block in a field for which the coefficient b is known.

A formulae (5), which may be used to estimate the most efficient size of plot for any given experiment, has also been deduced. The cost of using plots of other than the most efficient size is indicated graphically in Fig. 7.

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